Joint Institute for Nuclear Research
Bogoliubov Laboratory of Theoretical Physics

Isotropic oscillator in the space of constant positive curvature. Interbasis expansions.

Ye.M.Hakobyan [1], G.S.Pogosyan [2], A.N.Sissakian and S.I.Vinitsky

This work is devoted to the memory of
our dear friend I.V. Lutsenko

Abstract

The Schrödinger equation is thoroughly analysed for the isotropic oscillator in the three-dimensional space of constant positive curvature in the spherical and cylindrical systems of coordinates. The expansion coefficients between the spherical and cylindrical bases of the oscillator are calculated. It is shown that the relevant coefficients are expressed through the generalised hypergeometric functions ${}_4F_3$ of the unit argument or 6_j Racah symbols extended over their indices to the region of real values. Limiting transitions to a free motion and flat space are considered in detail.

Elliptic bases of the oscillator are constructed in the form of expansion over the spherical and cylindrical bases. The corresponding expansion coefficients are shown to obey the three-term recurrence relations.

[1]yera@thsun1.jinr.dubna.su
[2]pogosyan@thsun1.jinr.dubna.su

1 Introduction

Starting with the classical works by Schrödinger [1], Stivenson [3] and Infeld [2] systems with accidental degeneracy in spaces of constant curvature have attracted attention of many researchers in connection with nontrivial realization of hidden symmetry in these problems and with possible applications, especially to constructing many-particle wave functions [4], nonrelativistic models of quark systems [5] and solutions of the two-center problem [6].

Essential advances in the theory of systems with accidental degeneracy have been made by Nishino [7], Higgs [8], Leemon [9] and [10, 11, 12, 13]. It has been shown that the complete degeneracy of the spectrum of the Coulomb problem and harmonic oscillator on the three-dimensional sphere in the orbital and azimuthal quantum numbers is caused by an additional integral of motion: an analog of Runge–Lenz's vector (for the Coulomb potential) and an analog of Demkov's tensor (for the oscillator). However, in contrast with the flat space the integrals of motion for the Coulomb problem and isotropic oscillator do not form the Lie algebra as the relevant commutators are nonlinear. The latter does not allow one to restore, respectively, the algebra or the group of hidden symmetry. Later in the works [14,15] it has been shown that as an algebra of hidden symmetry one can use quadratic algebras of the general type, the so-called Racah algebras. Systems with hidden symmetry for the harmonic potential and those of Winternitz–Smorodinsky's type in the three-dimensional space of constant curvature were also studied by using the technique of path integrals in the papers by Barut, Inomata and Junker [16,17], Grosche [18] and Grosche et al. [19,20,21].

A possible way of determining a group of hidden symmetry of systems with accidental degeneracy is determination of the expansion coefficients between different bases obtained after the separation of variables in the Schrödinger equation. Such interbasis expansions have first been considered for the "sphere–cylinder" transitions (isotropic oscillator on the sphere), "sphere–parabola" transitions and those between spherical and elliptical bases (for the Coulomb potential on the sphere and hyperboloid) in [14,15,25]. It has been shown in [19] that like for the Helmholtz equation [22] variables in the Schrödinger equation for the potential of the isotropic oscillator on the three-dimensional sphere are separated into all the six orthogonal systems of coordinates: spherical, cylindrical, sphero-conical, two elliptic and ellipsoidal coordinate systems.

The aim of the present paper is the description of solutions of the Schrödinger equation in the spherical, cylindrical and two elliptic systems of coordinates and the calculation of expansion coefficients between the corresponding bases. Note that the solution of the Schrödinger equation for the isotropic oscillator in the spherical system of coordinates was found in [8,9,14] and in the cylindrical and elliptic systems of coordinates are presented for the first time.

The paper is organised as follows. Section 2 presents some known results related to the Schrödinger equation for the three-dimensional space of constant curvature. Section 3 is devoted to the solution of the Schrödinger equation for the potential of the isotropic oscillator in the spherical and cylindrical systems of coordinates. Section 4 is the calculation of coefficients of the interbasis expansion between spherical and cylindrical bases using the explicit expression for the wave functions of the isotropic oscillator. In Section 5 the elliptic bases of the isotropic oscillator are constructed as expansion over the spherical and cylindrical ones.

2 The Schrödinger equation and integrals of motion

The Schrödinger equation in the space of constant curvature has the form

$$\left[-\frac{\hbar^2}{2\mu}\Delta_{LB} + V(\mathbf{x})\right]\Psi = E\Psi. \tag{1}$$

where Δ_{LB} is the Laplace–Beltrami operator that in an arbitrary system of coordinates is given by

$$\Delta_{LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}g^{ik}\frac{\partial}{\partial x^k}, \qquad ds^2 = g_{ik}dx^i dx^k \tag{2}$$

$$g^{ik} = (g_{ik})^{-1}, \quad g = det(g_{ik}) \quad (i,k = 1,2,3).$$

Choosing a metric of the space of constant curvature in the form ($r^2 = x_i x_i$)

$$g_{ik} = \frac{1}{1 + r^2/R^2}\left[\left(\delta_{ik} - \frac{x_i x_k}{r^2}\right) + \frac{1}{1 + r^2/R^2}\frac{x_i x_k}{r^2}\right], \tag{3}$$

we derive the following expression for the Laplace–Beltrami operator:

$$\Delta_{LB} = \left(1 + \frac{r^2}{R^2}\right)\left[\left(\delta_{ik} + \frac{x_i x_k}{R^2}\right)\frac{\partial^2}{\partial x_i \partial x_k} + \frac{x_i}{R^2}\frac{\partial}{\partial x_i}\right] = -\left(P_i P_i + \frac{1}{R^2}L_i L_i\right) \tag{4}$$

where R is the curvature radius,

$$P_i = -i\left[\frac{\partial}{\partial x_i} + \frac{x_i x_k}{R^2}\frac{\partial}{\partial x_k}\right], \qquad L_i = -i\epsilon_{ijk}x_j\frac{\partial}{\partial x_k} = \epsilon_{ijk}x_i P_j \tag{5}$$

and the following commutation relations hold:

$$[P_i, x_l] = -i\left(\delta_{il} + \frac{x_i x_l}{R^2}\right), \qquad [L_i, x_l] = -i\epsilon_{ijl}x_j \tag{6}$$

It is easily seen that within the limits of the flat space, i.e. as $R \to \infty$, the operator P_i corresponds to an ordinary momentum operator, and the Laplace–Beltrami operator (2) turns into an ordinary Laplace operator in the flat three-dimensional space E_3.

The issue of generalising the problem of isotropic oscillator for the spaces of constant curvature with the use of the conformally flat metric in the classical mechanics has obviously been solved for the first time in [7], where, in particular, an additional integral of motion characteristic of an oscillator interaction was found. Later, in [8] it has been shown that if the metric of the curved space is chosen in the form of (3), the role of the potential of isotropic oscillator in the flat space is played by

$$V(r) = \frac{\mu\omega^2 r^2}{2}, \tag{7}$$

and an additional integral of motion has the form:

$$D_{ik} = \frac{1}{2}\left(P_i P_k + P_k P_i\right) + \frac{\mu^2\omega^2}{\hbar^2} x_i x_k, \tag{8}$$

which in the limit of large R exactly transforms into Demkov's tensor [23]. For the operators L_i and D_{ij} the following commutation relations are valid:

$$\begin{aligned}
[D_{ij}, L_k] &= i\left(\epsilon_{ikl} D_{jl} + \epsilon_{jkm} D_{im}\right), \\
[D_{ik}, D_{jl}] &= \frac{i\mu^2 w^2}{\hbar^2}\left(\delta_{li} L_{kj} + \delta_{lk} L_{ij} + \delta_{ij} L_{kl} + \delta_{jk} L_{il}\right) + \frac{i}{2R^2}\Big(\{L_{ij}, D_{lk}\} \\
&+ \{L_{il}, D_{kj}\} + \{L_{kj}, D_{il}\} + \{L_{kl}, D_{ij}\}\Big), \quad L_{ik} = x_i P_k - x_k P_i.
\end{aligned}$$

where $\{,\}$ means the anticommutator of two operators.

The three-dimensional space of constant positive curvature can also be realised geometrically on the three-dimensional sphere S_3 of the radius R, imbedded into the four-dimensional Euclidean space, i.e on the hypersurface

$$q_0^2 + q_i q_i = \mathbf{R}^2,$$

where the coordinates q_i change in the region $q_i q_i \leq R^2$ and to each value of q_i correspond two points on the sphere. Relation between the coordinates x_i in the tangent space and q_μ ($\mu = 0, 1, 2, 3$) is given by

$$q_i = \frac{x_i}{\sqrt{1 + r^2/R^2}}, \quad q_0 = \frac{R}{\sqrt{1 + r^2/R^2}}$$

obtained under the mapping from the center of the three-dimensional hypersphere onto the plane tangent to the "North pole". Such a parametrization of the space of constant curvature is often called in literature the "geodesic parametrization" [24] and in a one-to-one manner reflects only the hemisphere (in this case the upper one) or the sphere with identified diametrically opposite points.

In the coordinates q_μ we have

$$P_i = -\frac{1}{R} N_i = \frac{i}{R}\left(q_i \frac{\partial}{\partial q_0} - q_0 \frac{\partial}{\partial q_i}\right), \quad L_i = -i\epsilon_{ijk} q_j \frac{\partial}{\partial q_k} \tag{9}$$

and

$$\Delta_{LB} = -\frac{1}{R^2}\left(N_i^2 + L_i^2\right)$$

where the operators L_i N_i are generators of the group O(4)

$$[L_i, L_j] = i\epsilon_{ijk} L_k, \quad [L_i, N_j] = i\epsilon_{ijk} N_k, \quad [N_i, N_j] = i\epsilon_{ijk} L_k$$

The potential of the isotropic oscillator is given by the symmetric function

$$V(r) \equiv V(q) = \frac{\mu\omega^2}{2}\frac{q^2}{1-q^2/R^2}, \quad q^2 = q_1^2 + q_2^2 + q_3^2 \tag{10}$$

with respect to the upper and lower hemispheres, equals zero at the poles of the sphere and has a singularity at the equator. An additional integral of motion is given by the expression

$$D_{ik} = \frac{1}{2R^2}\left(N_i N_k + N_k N_i\right) + \frac{\mu^2\omega^2}{\hbar^2}\frac{q_i q_k}{1-q^2/R^2} \tag{11}$$

and like in the case of the flat space leads to separation of variables in the Schrödinger equation in more than one system of coordinates.

3 Solution of the Schrödinger equation

3.1 Spherical basis

In the spherical system of coordinates

$$\begin{array}{ll} q_1 = R\sin\chi\sin\vartheta\cos\varphi, & q_2 = R\sin\chi\sin\vartheta\sin\varphi, \\ q_3 = R\sin\chi\cos\vartheta, & q_0 = R\cos\chi, \end{array}$$

$$0 \le \chi \le \pi \ , \ 0 \le \vartheta \le \pi \ , \ 0 \le \varphi < 2\pi,$$

the oscillator potential has the form

$$V = \frac{\mu\omega^2 R^2}{2}\mathrm{tg}^2\chi.$$

Choosing the wave function according to

$$\Psi(\chi,\vartheta,\varphi;R) = \frac{1}{\sqrt{R^3}}Z(\chi)Y_{lm}(\vartheta,\varphi), \quad l \in \mathbf{N}, \quad m \in \mathbf{Z}, \tag{12}$$

where $Y_{lm}(\vartheta,\varphi)$ is an ordinary spherical function [31], after separation of variables in the Schrödinger equation we have

$$\left\{\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} + \frac{2\mu R^2}{\hbar^2}\left[E - \frac{\hbar^2}{2\mu R^2}\frac{l(l+1)}{\sin^2\chi} - \frac{\mu\omega^2 R^2}{2}tg^2\chi\right]\right\}Z(\chi;R) = 0 \tag{13}$$

Then, introducing the notation

$$\left(l+\frac{1}{2}\right)^2 = k_1^2, \quad \frac{\mu^2\omega^2 R^4}{\hbar^2} + \frac{1}{4} = k_2^2, \quad \frac{2\mu R^2 E}{\hbar^2} + \frac{\mu^2\omega^2 R^4}{\hbar^2} + 1 = \mathcal{E} \tag{14}$$

and making the substitution

$$Z(\chi) = \frac{f(\chi)}{\sin\chi},$$

we arrive at the equation without the first derivative of the Pöschl –Teller-type

$$\frac{d^2 f}{d\chi^2} + \left[\mathcal{E} - \frac{k_1^2 - \frac{1}{4}}{\sin^2\chi} - \frac{k_2^2 - \frac{1}{4}}{\cos^2\chi}\right] f = 0,$$

whose general solution is well known [35]. The requirement of regularity of the wave function $Z(\chi)$ at $\chi = 0$ and $\pi/2$ leads to quantization of the isotropic oscillator energy

$$E_N^{\nu}(R) = \frac{\hbar^2}{2\mu}\left[\frac{(N+1)(N+3)}{R^2} + \frac{2\nu}{R^2}\left(N + \frac{3}{2}\right)\right], \tag{15}$$

where the principal quantum number $N = 0, 1, ...$ is related with the radial and orbital quantum numbers by $N = 2n_r + l$, and the following notation is introduced:

$$\nu \equiv k_2 - \frac{1}{2} = \frac{1}{2}\sqrt{1 + \frac{4\mu^2\omega^2 R^4}{\hbar^2}} - \frac{1}{2}.$$

Note that the degree of degeneracy, like in the case of motion in the field of the harmonic isotropic oscillator in the three-dimensional Euclidean space, is equal to $(N+1)(N+2)/2$.

The solution of the quasiradial Schrödinger equation (13), orthonormalised in the interval $\chi \in [0, \frac{\pi}{2}]$ is

$$\begin{aligned} Z(\chi) &\equiv Z_{Nl}^{\nu}(\chi) \\ &= \sqrt{\frac{2(N+\nu+2)(\frac{N-l}{2})!\Gamma(\frac{N+l}{2}+\nu+2)}{\Gamma(\frac{N+l+3}{2})\Gamma(\frac{N-l+3}{2}+\nu)}}(\sin\chi)^l(\cos\chi)^{\nu+1}P_{\frac{N-l}{2}}^{(l+\frac{1}{2},\nu+\frac{1}{2})}(\cos 2\chi), \end{aligned} \tag{16}$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials.

Let us consider the limit of the flat space. It is easily seen that at large R ($\nu \to \lambda R^2$, $\lambda = \mu\omega/\hbar^2$) the formula (14) is used to restore the formula for the energy spectrum of the three-dimensional oscillator

$$\lim_{R\to\infty} E_N^{\nu}(R) = E_N = \hbar\omega\left(N + \frac{3}{2}\right)$$

Transition from the spherical system of coordinates on S_3 to the relevant system of coordinates on E_3 is accomplished within the limit $R \to \infty$, $\chi \to 0$ and $\chi \sim r/R$ where r is the radius vector in the three-dimensional flat space [26]. Using the known relation for the Jacobi polynomials [28]

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right) = L_n^{\alpha}(x), \tag{17}$$

where $L_n^{\alpha}(x)$ are the Laguerre polynomials, and taking account of the limiting relations

$$\sqrt{\frac{2(N+\nu+2)\Gamma(\frac{N+l}{2}+\nu+2)}{\Gamma(\frac{N-l+3}{2}+\nu)}}\ \frac{(\sin\chi)^l}{\sqrt{R^3}} \Rightarrow \sqrt{2\lambda^{3/2}}(\sqrt{\lambda}r)^l, \ \ (\cos\chi)^{\nu+1/2} \Rightarrow e^{-\frac{\lambda r^2}{2}},$$

we immediately get that

$$\lim_{R\to\infty} \frac{1}{\sqrt{R^3}} Z^{\nu}_{Nl}(\chi) = \left(\frac{\lambda}{\pi}\right)^{1/4} \sqrt{\frac{2^{l+1}\lambda(N-l)!!}{(N+l+1)!!}}\, (\sqrt{\lambda} r)^l\, e^{-\frac{\lambda r^2}{2}} L^{l+\frac{1}{2}}_{\frac{N-l}{2}}(\lambda r^2) = R_{Nl}(r), \tag{18}$$

where $R_{Nl}(r)$ is the orthonormalised spherical radial wave function of an ordinary three-dimensional isotropic oscillator in flat space [35].

The second interesting limit is the transition to a free motion. As $\nu \to 0$ ($\omega \to 0$) we have

$$\lim_{\nu\to 0} E^{\nu}_N(R) = E^0_N(R) = \frac{\hbar^2}{2\mu} \frac{(N+1)(N+3)}{R^2}$$

Comparing the above-derived expression with the formula for the energy of a free motion of particles on the sphere $\frac{\hbar^2 J(J+2)}{2\mu R^2}$ we get that $J = N+1$ and $J = 1, 2, ...$, and consequently, the ground state with $J = 0$ is missing in the limiting spectrum. As for the oscillator spectrum $(N-l)$ is always even, within the limit of a free motion $(J-l)$ takes odd values and at fixed J there exist only states with $l = J-1, J-3, ...$ and, correspondingly, the degree of degeneracy of the limiting spectrum is smaller than $(J+1)^2$, as it should be for the free motion on the sphere.

Further, using the transformation [34]

$$x\, P^{(\lambda-\frac{1}{2},\frac{1}{2})}_m(2x^2-1) = \frac{\Gamma(\lambda)\Gamma(m+3/2)}{\Gamma(\lambda+m+1)\Gamma(1/2)}\, C^{\lambda}_{2m+1}(x), \tag{19}$$

connecting odd Gegenbauer polynomials with the Jacobi polynomials and passing from the quantum number N to J, we come to the function

$$\lim_{\nu\to 0} Z^{\nu}_{Nl}(\chi) = \frac{2^{l+1} l!}{\sqrt{\pi}} \sqrt{\frac{(J+1)(J-l)!}{(J+l+1)!}} (\sin\chi)^l C^{l+1}_{J-l}(\cos\chi),$$

which, with an accuracy to a factor, $\sqrt{2}$ corresponds to the solution of the free Schrödinger equation on the three-dimensional sphere with the impenetrable barrier at the equator ($\chi = \pi/2$).

3.2 Cylindrical basis

In the cylindrical system of coordinates

$$\begin{array}{ll} q_1 = R\sin\alpha\cos\phi_1, & q_2 = R\sin\alpha\sin\phi_1, \\ q_3 = R\cos\alpha\sin\phi_2, & q_0 = R\cos\alpha\cos\phi_2, \end{array}$$

$$0 \le \alpha \le \pi/2,\ 0 \le \phi_1 < 2\pi,\ -\pi \le \phi_2 \le \pi,$$

the potential of the isotropic oscillator is written as

$$V = \frac{\mu\omega^2 R^2}{2} \left[\frac{1}{\cos^2\alpha\cos^2\phi_2} - 1\right]. \tag{20}$$

Choosing the wave function in the form

$$\Psi(\phi_1,\alpha,\phi_2;R)=\frac{1}{\sqrt{R^3}}\,\Phi(\alpha)\,K(\phi_2)\,\frac{e^{im\phi_1}}{\sqrt{2\pi}},$$

after separation of variables we arrive at two differential equations of the Pöschl–Teller-type

$$\frac{d^2M}{d\alpha^2}+\left(\mathcal{E}-\frac{m^2-\frac14}{\sin^2\alpha}-\frac{A-\frac14}{\cos^2\alpha}\right)M=0, \tag{21}$$

$$\frac{d^2K}{d\phi^2}+\left(4A-\frac{k_2^2-\frac14}{\sin^2\phi}+\frac{k_2^2-\frac14}{\cos^2\phi}\right)K=0, \tag{22}$$

where $(\sin\alpha\cos\alpha)^{-1/2}M(\alpha)=\Phi(\alpha)$ $\phi=\frac{\phi_2}{2}+\frac{\pi}{4}$, $\phi\in[0,\frac{\pi}{2}]$ $\mathcal{E}$ is determined by expression (14). The spectrum of constants is determined by:

$$A=(n_3+\nu+1)^2,\qquad \mathcal{E}=(2n+n_3+|m|+\nu+2)^2,$$

where the quantum numbers n_3 and n run the values $0,1,2,....$ Assuming the principal quantum number N to be equal to $N=2n+|m|+n_3$, we get the formula (15) for the isotropic oscillator energy. For the cylindrical basis we get the following expression:

$$\Psi(\phi_1,\alpha,\phi_2;R)\equiv\Psi^{\nu}_{Nmn_3}(\phi_1,\alpha,\phi_2;R)=\frac{1}{\sqrt{R^3}}\Phi^{\nu}_{N|m|n_3}(\alpha)K^{\nu}_{n_3}(\phi_2)\frac{e^{im\phi_1}}{\sqrt{2\pi}} \tag{23}$$

where the functions $K^{\nu}_{n_3}(\phi_2)$ and $\Phi^{\nu}_{N|m|n_3}(\alpha)$, normalised in the interval $\phi_2\in[-\frac{\pi}{2},\frac{\pi}{2}]$, $\alpha\in[0,\frac{\pi}{2}]$, are

$$\begin{aligned}
\Phi^{\nu}_{N|m|n_3}(\alpha) &= \sqrt{\frac{2(N+\nu+2)(\frac{N-|m|-n_3}{2})!\Gamma(\frac{N+|m|+n_3}{2}+\nu+2)}{(\frac{N+|m|-n_3}{2})!\Gamma(\frac{N-|m|+n_3}{2}+\nu+2)}}\\
&\times\ (\sin\alpha)^{|m|}(\cos\alpha)^{n_3+\nu+1}P^{(|m|,\,n_3+\nu+1)}_{\frac{N-|m|-n_3}{2}}(\cos 2\alpha),\\
K^{\nu}_{n_3}(\phi_2) &= \frac{\sqrt{(n_3+\nu+1)\Gamma(n_3+2\nu+2)(n_3)!}}{2^{\nu+\frac12}\Gamma(n_3+\nu+\frac32)}(\cos\phi_2)^{\nu+1}P^{(\nu+\frac12,\nu+\frac12)}_{n_3}(\sin\phi_2).
\end{aligned} \tag{24}$$

Note that to a cylindrical system of coordinates corresponds an additional integral of motion

$$M=-\frac{d^2}{d{\phi_2}^2}+\frac{\mu^2\omega^2R^4}{\hbar^2}\frac{1}{\cos^2\phi_2}=R^2D_{33}+\frac{\mu^2\omega^2R^4}{\hbar^2} \tag{25}$$

Within large R the cylindrical system of coordinates on the sphere turns into an ordinary cylindrical system of coordinates (ρ,ϕ,z) in the Euclidean space E_3 [26]. Passing to the limit $R\to\infty$ and $\alpha,\phi_2\to 0$, and assuming

$$\sin\alpha\sim\alpha\sim\frac{\rho}{R},\qquad \sin\phi_2\sim\phi_2\sim\frac{z}{R},$$

as well as using formula (17) and the following relation [28]

$$\lim_{\lambda\to\infty} \lambda^{-n/2} C_n^{\lambda/2}\left(t\sqrt{\frac{2}{\lambda}}\right) = \frac{2^{-n/2}}{n!}\mathcal{H}_n(t),$$

where $\mathcal{H}_n(z)$ are the Hermite polynomials [28], we get

$$\begin{aligned}
\lim_{R\to\infty}\frac{1}{R}\Phi^{\nu}_{N|m|n_3}(\alpha) &= \sqrt{\frac{2\lambda(\frac{N-|m|-n_3}{2})!}{(\frac{N+|m|-n_3}{2})!}}e^{-\frac{\lambda\rho^2}{2}}(\sqrt{\lambda}\rho)^{|m|}L^{|m|}_{\frac{N-|m|-n_3}{2}}(\lambda\rho^2),\\
\lim_{R\to\infty}\frac{1}{\sqrt{R}}K^{\nu}_{n_3}(\phi_2) &= \left(\frac{\lambda}{\pi}\right)^{1/4}\frac{e^{-\frac{\lambda z^2}{2}}}{\sqrt{2^{n_3}(n_3)!}}\mathcal{H}_{n_3}(\sqrt{\lambda}z).
\end{aligned}$$

Thus, formula (23) leads to the orthonormalised cylindrical basis of the isotropic oscillator in the flat space.

Within the limit of a free motion $\nu\to 0$ $(\omega\to 0)$ with the use of the formula [28]

$$P_n^{(\frac{1}{2},\frac{1}{2})}(x) = \frac{(2n+1)!!}{2^n(n+1)!}\,\frac{\sin[(n+1)\arccos x]}{\sin(\arccos x)},$$

we have

$$\begin{aligned}
\lim_{\nu\to 0} K^{\nu}_{n_3}(\phi_2) &= \sqrt{\frac{2}{\pi}}\,\sin\{(n_3+1)(\phi_2+\pi/2)\},\\
\lim_{\nu\to 0}\Phi^{\nu}_{N|m|n_3}(\alpha) &= \sqrt{\frac{(N+2)(\frac{N-n_3-|m|}{2})!(\frac{N+n_3+|m|}{2}+1)!}{(\frac{N-|m|+n_3}{2}+1)!(\frac{N+|m|-n_3}{2})!}}\\
&\cdot\quad (\sin\alpha)^{|m|}(\cos\alpha)^{n_3+1}P^{(|m|,\,n_3+1)}_{\frac{N-n_3-|m|}{2}}(\cos 2\alpha).
\end{aligned}$$

Assuming $n_3+1=|m_2|$, $|m|=|m_1|$ and $J=N+1$, we obtain (with an accuracy to a factor $\sqrt{2}$) odd solutions of the Schrödinger equation for a free motion in the cylindrical system of coordinates.

4 Expansion between spherical and cylindrical bases

4.1 Calculation of the transition coefficients

Let us write the expansion between the spherical and cylindrical bases of the isotropic oscillator in the form

$$\Psi^{\nu}_{Nlm}(\chi,\vartheta,\varphi;R) = \sum_{n_3=0,1}^{N-|m|} W^{n_3}_{Nlm}(\nu)\Psi^{\nu}_{Nmn_3}(\phi_1,\alpha,\phi_2;R), \tag{26}$$

where the quantum number n_3 takes even and odd values depending on parity $N-|m|$.

To calculate an explicit form of the expansion coefficients $W_{Nlm}^{n_3}(\nu)$ it is sufficient to use orthogonality in one of the variables for the functions entering into the cylindrical wave function and to fix at the most appropriate point the second variable that does not participate in integration. Passing beforehand in the left-hand side of the expansion (26) from the spherical coordinates to the cylindrical ones, according to the formulae

$$\cos\chi = \cos\alpha \cdot \cos\phi_2, \quad \sin\vartheta = \frac{\sin\alpha}{\sqrt{1-\cos^2\alpha\cos^2\phi_2}}, \quad \phi = \phi_1$$

and taking into account that as $\alpha \to 0$

$$\cos\chi \to \cos\phi_2, \quad \sin\vartheta \to \frac{\sin\alpha}{\sin\phi_2} \to 0,$$

we derive

$$Z_{Nl}^{\nu}(\chi) \to Z_{Nl}^{\nu}(\phi_2)$$

$$Y_{lm}(\vartheta,\phi) \to \frac{(-1)^{\frac{m+|m|}{2}}}{2^{|m|}|m|!}\sqrt{\frac{2l+1}{2}\frac{(l+|m|)!}{(l-|m|)!}}\frac{(\sin\alpha)^{|m|}}{(\sin\phi_2)^{|m|}}\frac{e^{im\phi}}{\sqrt{2\pi}}$$

$$\Phi_{N|m|n_3}^{\nu}(\alpha) \to \sqrt{\frac{2(N+\nu+2)(\frac{N+|m|-n_3}{2})!\Gamma(\frac{N+|m|+n_3}{2}+\nu+2)}{(\frac{N-|m|-n_3}{2})!\Gamma(\frac{N-|m|+n_3}{2}+\nu+2)}}\frac{(\sin\alpha)^{|m|}}{|m|!}.$$

Then, substituting the asymptotic formulae derived into the interbasis expansion (26), reducing $(\sin\alpha)^{|m|}$ and using the orthogonality of the functions $K_{n_3}^{\nu}(\phi_2)$ in the interval $-\frac{\pi}{2} \le \phi_2 \le \frac{\pi}{2}$, we arrive at the following integral representation for the coefficients $W_{Nlm}(\nu)$:

$$\begin{aligned} W_{Nlmn_3}(\nu) &= (-1)^{\frac{m+|m|}{2}}\sqrt{\frac{(2l+1)(n_3+\nu+1)}{2^{|m|+\nu+1}\Gamma(n_3+\nu+\frac{3}{2})}\frac{(l+|m|)!(\frac{N-|m|-n_3}{2})!}{(l-|m|)!(\frac{N+|m|-n_3}{2})!}\frac{(n_3)!(\frac{N-l}{2})!}{\Gamma(\frac{N+l}{2}+\frac{3}{2})}} \\ &\cdot \sqrt{\frac{\Gamma(\frac{N-|m|+n_3}{2}+\nu+2)\Gamma(\frac{N+l}{2}+\nu+2)\Gamma(n_3+2\nu+2)}{\Gamma(\frac{N+|m|+n_3}{2}+\nu+2)\Gamma(\frac{N-l}{2}+\nu+\frac{3}{2})}}\; A_{N|m|n_3}^{l}(\nu), \end{aligned} \tag{27}$$

where

$$A_{N|m|n_3}^{l}(\nu) = \int_{-\pi/2}^{\pi/2}(\sin\phi_2)^{l-|m|}(\cos\phi_2)^{2\nu+2}P_{\frac{N-l}{2}}^{(l+\frac{1}{2},\nu+\frac{1}{2})}(\cos 2\phi_2)P_{n_3}^{(\nu+\frac{1}{2},\nu+\frac{1}{2})}(\sin\phi_2)d\phi_2. \tag{28}$$

A complete solution of the problem needs calculation of the integral in formula (27). Let us consider separately the cases of even and odd quantum number n_3. Separating the interval of integration (28) into two intervals $(-\frac{\pi}{2},0)$ and $(0,\frac{\pi}{2})$, after the substitution in the first integral

$\phi_2 \to -\phi_2$ we see that the value of the integral is just doubled due to parity $(l-|m|-n_3)$. Then, using the well-known transformation for the Jacobi polynomials [34]

$$P_{n_3}^{(\alpha,\alpha)}(x) = \begin{cases} \dfrac{\Gamma(n_3+\alpha+1)(\frac{n_3}{2})!}{\Gamma(\frac{n_3}{2}+\alpha+1)(n_3)!}\; P_{\frac{n_3}{2}}^{(\alpha,-\frac{1}{2})}(2x^2-1) & \text{for } n_3 - \text{even}, \\ \\ \dfrac{\Gamma(n_3+\alpha+1)\left(\frac{n_3-1}{2}\right)!}{\Gamma\left(\frac{n_3+1}{2}+\alpha\right)(n_3)!}\, x\, P_{\frac{n_3-1}{2}}^{(\alpha,\frac{1}{2})}(2x^2-1) & \text{for } n_3 - \text{odd}, \end{cases}$$

after the substitution $x = \cos^2\phi_2$, we come to the following two table integrals for even and odd n_3

$$\begin{aligned} A_{N|m|n_3}^{l(+)}(\nu) &= \frac{(-1)^{\frac{n_3}{2}}}{2^{\nu+\frac{l-|m|}{2}+1}} \cdot \frac{\Gamma(n_3+\nu+\frac{3}{2})(\frac{n_3}{2})!}{\Gamma(\frac{n_3+3}{2}+\nu)(n_3)!} \times \\ &\times \int_{-1}^{1}(1-x)^{\frac{l-|m|-1}{2}}(1+x)^{\nu+\frac{1}{2}} P_{\frac{N-l}{2}}^{(l+\frac{1}{2},\nu+\frac{1}{2})}(x) P_{n_3}^{(-\frac{1}{2},\nu+\frac{1}{2})}(x)dx, \\ A_{N|m|n_3}^{l(-)}(\nu) &= \frac{(-1)^{\frac{n_3-1}{2}}}{2^{\nu+\frac{l-|m|+3}{2}}} \cdot \frac{\Gamma(n_3+\nu+\frac{3}{2})\left(\frac{n_3-1}{2}\right)!}{\Gamma\left(\frac{n_3+2}{2}+\nu\right)(n_3)!} \\ &\times \int_{-1}^{1}(1-x)^{\frac{l-|m|}{2}}(1+x)^{\nu+\frac{1}{2}} P_{\frac{N-l}{2}}^{(l+\frac{1}{2},\nu+\frac{1}{2})}(x) P_{\frac{n_3-1}{2}}^{(\frac{1}{2},\nu+\frac{1}{2})}(x)dx. \end{aligned}$$

Using the formula for integration of the two Jacobi polynomials[30]

$$\int_{-1}^{1}(1-x)^{\tau}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\rho,\beta)}(x)dx = \frac{2^{\beta+\tau+1}\Gamma(\alpha-\tau+n)\Gamma(\beta+n+l)}{(m)!(n)!\Gamma(\rho+1)\Gamma(\alpha-\tau)}$$
$$\frac{\Gamma(\rho+m+1)\Gamma(\tau+1)}{\Gamma(\beta+\tau+n+2)}\,{}_4F_3\left\{\begin{matrix} -m,\ \rho+\beta+m+1,\ \tau+1,\ \tau-\alpha+1 \\ \rho+1,\ \beta+\tau+n+2,\ \tau-\alpha-n+1 \end{matrix}\middle| 1\right\},$$

we immediately get that $A_{N|m|n_3}^{l(\pm)}(\nu)$ is expressed through the generalised hypergeometric function ${}_4F_3$ of the unit argument

$$\begin{aligned} A_{N|m|n_3}^{l(+)}(\nu) &= \frac{(-1)^{\frac{n_3}{2}}}{\sqrt{\pi}} \frac{\Gamma(n_3+\nu+\frac{3}{2})\Gamma(\frac{N+|m|}{2}+1)\Gamma(\frac{N-l+3}{2}+\nu)}{(n_3)!\Gamma(\frac{n_3+3}{2}+\nu)(\frac{N-l}{2})!\Gamma(\frac{l+|m|}{2}+1)} \frac{\Gamma(\frac{l-|m|+1}{2})\Gamma(\frac{n_3+1}{2})}{\Gamma(\frac{N-|m|}{2}+\nu+2)} \\ &\cdot\ {}_4F_3\left\{\begin{matrix} -\frac{n_3}{2},\ \frac{n_3}{2}+\nu+1,\ \frac{l-|m|+1}{2},\ -\frac{l+|m|}{2} \\ \frac{1}{2},\ \frac{N-|m|}{2}+\nu+2,\ -\frac{N+|m|}{2} \end{matrix}\middle| 1\right\}, \end{aligned}$$

and analogously

$$\begin{aligned} A_{N|m|n_3}^{l(-)}(\nu) &= \frac{(-1)^{\frac{n_3-1}{2}}}{\sqrt{\pi}} \frac{2\Gamma(n_3+\nu+\frac{3}{2})\Gamma(\frac{N+|m|+1}{2})\Gamma(\frac{N-l+3}{2}+\nu)}{(n_3)!\Gamma(\frac{n_3+2}{2}+\nu)(\frac{N-l}{2})!\Gamma(\frac{l+|m|+1}{2})} \frac{\Gamma(\frac{l-|m|}{2}+1)\Gamma(\frac{n_3}{2}+1)}{\Gamma(\frac{N-|m|+5}{2}+\nu)} \\ &\cdot\ {}_4F_3\left\{\begin{matrix} -\frac{n_3-1}{2},\ \frac{n_3+3}{2}+\nu,\ \frac{l-|m|}{2}+1,\ -\frac{l+|m|-1}{2} \\ \frac{3}{2},\ \frac{N-|m|+5}{2}+\nu,\ -\frac{N+|m|-1}{2} \end{matrix}\middle| 1\right\}. \end{aligned}$$

Taking account of the known symmetry property for the series ${}_4F_3(1)$ of the Saalschütz-type [29]

$$
{}_4F_3\left\{\begin{matrix} -n,\ b,\ c,\ d \\ e,\ f,\ g \end{matrix}\middle|\, 1\right\} = \frac{(f-b)_n(g-b)_n}{(f)_n(g)_n}\,{}_4F_3\left\{\begin{matrix} -n,\ b,\ e-c,\ e-d \\ e,\ b-f-n+1,\ b-g-n+1 \end{matrix}\middle|\, 1\right\},
$$

$$
-n+b+c+d = 1+e+f+g,
$$

one can easily be convinced that both the hypergeometric functions ${}_4F_3(1)$ entering into $A^{l(\pm)}_{N|m|n_3}(\nu)$ can be transformed to a unique form:

$$
{}_4F_3\left\{\begin{matrix} -\frac{n_3}{2},\ \frac{n_3}{2}+\nu+1,\ \frac{l-|m|+1}{2},\ -\frac{l+|m|}{2} \\ \frac{1}{2},\ \frac{N-|m|}{2}+\nu+2,\ -\frac{N+|m|}{2} \end{matrix}\middle|\, 1\right\} = \frac{(-1)^{\frac{n_3}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{l-|m|}{2}+1)\Gamma(\frac{l+|m|}{2}+1)}{\Gamma(\frac{n_3+1}{2})\Gamma(\frac{l+|m|-n_3}{2}+1)\Gamma(\frac{l-|m|-n_3}{2}+1)}
$$

$$
\frac{\Gamma(\frac{n_3}{2}+\nu+\frac{3}{2})\Gamma(\frac{N-|m|}{2}+\nu+2)(\frac{N+|m|-n_3}{2})!}{\Gamma(\frac{N+|m|}{2}+1)\Gamma(\frac{N-|m|+n_3}{2}+\nu+2)\Gamma(\nu+1)}\cdot {}_4F_3\left\{\begin{matrix} -\frac{n_3}{2},\ -\frac{n_3-1}{2},\ -\frac{N-l}{2},\ \frac{N+l}{2}+\nu+2 \\ \nu+\frac{3}{2},\ \frac{l+|m|-n_3}{2}+1,\ \frac{l-|m|-n_3}{2}+1 \end{matrix}\middle|\, 1\right\},
$$

$$
{}_4F_3\left\{\begin{matrix} -\frac{n_3-1}{2},\ \frac{n_3+3}{2}+\nu,\ \frac{l-|m|}{2}+1,\ -\frac{l+|m|-1}{2} \\ \frac{3}{2},\ \frac{N-|m|+5}{2}+\nu,\ -\frac{N+|m|-1}{2} \end{matrix}\middle|\, 1\right\} = \frac{(-1)^{\frac{n_3-1}{2}}\Gamma(\frac{3}{2})\Gamma(\frac{l-|m|+1}{2})\Gamma(\frac{l+|m|+1}{2})}{\Gamma(\frac{n_3}{2}+1)\Gamma(\frac{l+|m|-n_3}{2}+1)\Gamma(\frac{l-|m|-n_3}{2}+1)}
$$

$$
\frac{\Gamma(\frac{n_3}{2}+\nu+\frac{3}{2})\Gamma(\frac{N-|m|}{2}+\nu+\frac{5}{2})(\frac{N+|m|-n_3}{2})!}{\Gamma(\frac{N+|m|+1}{2})\Gamma(\frac{N-|m|+n_3}{2}+\nu+2)\Gamma(\nu+1)}\cdot {}_4F_3\left\{\begin{matrix} -\frac{n_3}{2},\ -\frac{n_3-1}{2},\ -\frac{N-l}{2},\ \frac{N+l}{2}+\nu+2 \\ \nu+\frac{3}{2},\ \frac{l+|m|-n_3}{2}+1,\ \frac{l-|m|-n_3}{2}+1 \end{matrix}\middle|\, 1\right\}.
$$

After simple transformations we finally derive the sought formula for the coefficients of the interbasis expansion $W^{n_3}_{Nlm}(\nu)$

$$
W^{n_3}_{Nlmn_3}(\nu) = \frac{(-1)^{\frac{m+|m|}{2}}\sqrt{\pi}}{2^{l+\nu+1}}\frac{\sqrt{(2l+1)(n_3+\nu+1)(l+|m|)!(l-|m|)!}}{\Gamma(\frac{l-|m|-n_3}{2}+1)\Gamma(\frac{l+|m|-n_3}{2}+1)\Gamma(\nu+\frac{3}{2})}
$$

$$
\cdot\sqrt{\frac{\Gamma(\frac{N+l}{2}+\nu+2)\Gamma(\frac{N-l+3}{2}+\nu)\Gamma(n_3+2\nu+2)(\frac{N-|m|-n_3}{2})!(\frac{N+|m|-n_3}{2})!}{\Gamma(\frac{N+l}{2}+\frac{3}{2})(\frac{N-l}{2})!\Gamma(\frac{N-|m|+n_3}{2}+\nu+2)\Gamma(\frac{N+|m|+n_3}{2}+\nu+2)(n_3)!}}
$$

$$
\cdot\,{}_4F_3\left\{\begin{matrix} -\frac{n_3}{2},\ -\frac{n_3-1}{2},\ -\frac{N-l}{2},\ \frac{N+l}{2}+\nu+2 \\ \nu+\frac{3}{2},\ \frac{l+|m|-n_3}{2}+1,\ \frac{l-|m|-n_3}{2}+1 \end{matrix}\middle|\, 1\right\}. \tag{29}
$$

Note that the expression we have derived for $W^{n_3}_{Nlm}(\nu)$ is independent of parity of the quantum number n_3.

4.2 Connection with the Racah coefficients

The interbasis expansion coefficients (29) can also be expressed through $6j$, the symbols or Racah coefficients of the SU(2) group, extended over their indices to the region of real values. Comparing the expression for $W^{l}_{Nmn_3}(\nu)$ with the representation of the Racah coefficients

$W(abed;\ cf)$ through the hypergeometric functions ${}_4F_3(1)$ of the unit argument [31]

$$W(abed;\ cf) = \frac{\Delta(abc)\Delta(cde)\Delta(aef)\Delta(bdf)}{(a+b-c)!(d+e-c)!(a-f+e)!(b-f+d)!(c-a-d+f)!}$$

$$\frac{(a+b+d+e+1)!}{(c-b-e+f)!}{}_4F_3\left\{ \begin{matrix} -a-b+c,\ -b-d+f,\ -a-e+f,\ c-d-e \\ -a-b-d-e-1,\ -a+c-d+f+1,\ -b+c-e+f+1 \end{matrix} \middle| 1 \right\}, \quad (30)$$

where $\Delta(abc)$ is

$$\Delta(abc) = \sqrt{\frac{(a+b-c)!(a-b+c)!(b+c-a)!}{(a+b+c+1)!}},$$

and taking account of the symmetry property of the Racah coefficients

$$W(abed;\ cf) = i(-1)^{a+b-c}W(ab\bar{e}\bar{d};\ c\bar{f}),$$

$$\bar{e} = -e-1, \ \ \bar{d} = -d-1, \ \ \bar{f} = -f-1,$$

after simple calculations we get the required formula

$$W_{Nlm}^{n_3}(\nu) = (-1)^{\frac{N-l}{2}+\frac{m+|m|}{2}}\sqrt{(l+1/2)(n_3+\nu+1)}W(abed;\ cf), \quad (31)$$

$$\begin{array}{lll} a = \frac{N+|m|}{4}, & b = \frac{N-|m|-1}{4}, & c = \frac{2l-1}{4}, \\ d = \frac{N-|m|}{4} + \frac{\nu}{2} + \frac{1}{4}, & e = \frac{N+|m|}{4} + \frac{\nu}{2}, & f = \frac{n_3}{2} + \frac{\nu}{2}. \end{array}$$

Then, using the relation of orthonormalization for the Racah coefficients [31]

$$\sum_c \sqrt{(2c+1)(2f+1)}W(abed;\ cf)W(abed;\ cf') = \delta_{ff'},$$

we can write the inverse expansion in the form

$$\Psi_{Nmn_3}^{\nu}(\phi_1, \alpha, \phi_2; R) = \sum_{l=|m|,|m|+1}^{N} \tilde{W}_{Nmn_3}^{l}(\nu)\Psi_{Nlm}^{\nu}(\chi, \vartheta, \varphi; R),$$

where summation over l starts with $|m|$ or $|m|+1$ depending on the parity of the number $N-|m|$, and the coefficients

$$\tilde{W}_{Nmn_3}^{l}(\nu) = W_{Nlm}^{n_3}(\nu)$$

can be expressed through the polynomials ${}_4F_3(1)$ with the use of the representation (30).

4.3 Limiting relations

Consider limiting transitions to the flat space and free motion in the expansion coefficients $W^l_{Nmn_3}(\nu)$.

4.3.1 As $R \to \infty$ the generalised hypergeometric function ${}_4F_3(1)$ transforms into ${}_3F_2(1)$ according to

$$ {}_4F_3\left\{ \begin{matrix} -\frac{n_3}{2}, & -\frac{n_3-1}{2}, & -\frac{N-l}{2}, & \frac{N+l}{2}+\nu+2 \\ \nu+1, & \frac{l+|m|-n_3}{2}+1, & \frac{l-|m|-n_3}{2}+1 & \end{matrix} \middle| 1 \right\} \Rightarrow {}_3F_2\left\{ \begin{matrix} -\frac{n_3}{2}, & -\frac{n_3-1}{2}, & -\frac{N-l}{2} \\ \frac{l+|m|-n_3}{2}+1, & \frac{l-|m|-n_3}{2}+1 & \end{matrix} \middle| 1 \right\} $$

Having made the relevant limiting transition in gamma functions, after simple algebraic transformations we get the known formula for the coefficients of the interbasis expansion between the spherical and cylindrical bases of the harmonic isotropic oscillator in the flat Euclidean space [33]:

$$ \lim_{\nu\to\infty} W^{n_3}_{Nlm}(\nu) = \frac{(-1)^{\frac{m+|m|}{2}}}{2^{l-n_3}} \sqrt{\frac{(N-|m|-n_3)!!(N+|m|-n_3)!!}{(N+l+1)!!(N-l)!!(n_3)!}} $$

$$ \frac{\sqrt{(2l+1)(l+|m|)!(l-|m|)!}}{\Gamma(\frac{l-|m|-n_3}{2}+1)\Gamma(\frac{l+|m|-n_3}{2}+1)}\, {}_3F_2\left\{ \begin{matrix} -\frac{n_3}{2}, & -\frac{n_3-1}{2}, & -\frac{N-l}{2} \\ \frac{l+|m|-n_3}{2}+1, & \frac{l-|m|-n_3}{2}+1 & \end{matrix} \middle| 1 \right\}. $$

On the other hand, the limiting transition to the flat space can directly be traced in the formula (31). Indeed, using at large R the asymptotic coupling [31]

$$ W(abe+R, d+R; c, f+R) \approx \frac{1}{\sqrt{2R(2c+1)}}\, C^{c,\, d-e}_{a,\, f-e;\, b,\, d-f} $$

and the symmetry property of the Clebsch–Gordan coefficients [31]

$$ C^{c,-\gamma}_{a,-\alpha;b,-\beta} = (-1)^{a+b-c}\, C^{c,\gamma}_{a,\alpha;\, b,\beta}, $$

we obtain that

$$ \lim_{\nu\to\infty} W^{n_3}_{Nlm}(\nu) = (-1)^{\frac{m+|m|}{2}}\, C^{\frac{2l-1}{4},\frac{2|m|-1}{4}}_{\frac{N+|m|}{4},\frac{N+|m|-2n_3}{4};\frac{N-|m|-1}{4},\frac{2n_3-N+|m|-1}{4}}. $$

4.3.2. Assuming $\nu = 0$, $(\omega = 0)$ and passing to the quantum numbers corresponding to the free motion on the sphere $m = m_1, n_3+1 = |m_2|, N = J-1$ (note that $J-l$ is odd and $J-|m_1|-|m_2|$ is even), we have

$$ \lim_{\nu\to 0} W^{n_3}_{Nlm}(\nu) = (-1)^{\frac{m_1+|m_1|}{2}} \frac{|m_2|}{2^l} \sqrt{\frac{(\frac{J+|m_1|-|m_2|}{2})!(\frac{J-|m_1|-|m_2|}{2})!(J+l+1)!!(J-l)!!}{(\frac{J+|m_1|+|m_2|}{2})!(\frac{J-|m_1|+|m_2|}{2})!(J+l)!!(J-l-1)!!}} $$

$$ \frac{\sqrt{(l+1/2)(l+|m|)!(l-|m|)!}}{\Gamma(\frac{l-|m_1|-|m_2|}{2}+\frac{3}{2})\Gamma(\frac{l+|m_1|-|m_2|}{2}+\frac{3}{2})} \cdot {}_4F_3\left\{ \begin{matrix} -\frac{|m_2|-1}{2}, & -\frac{|m_2|-2}{2}, & -\frac{J-l-1}{2}, & \frac{J+l+3}{2} \\ \frac{3}{2}, & \frac{l+|m|-|m_2|}{2}+\frac{3}{2}, & \frac{l-|m|-|m_2|}{2}+\frac{3}{2} & \end{matrix} \middle| 1 \right\} $$

$$ = (-1)^{\frac{J-l-1}{2}+\frac{m_1+|m_1|}{2}} \sqrt{(l+1/2)|m_2|}\, W(abed;\ cf), \qquad (32) $$

where

$$\begin{array}{lll} a=\frac{J+|m|-1}{4}, & b=\frac{J-|m|-2}{4}, & c=\frac{2l-1}{4}, \\ d=\frac{J-|m|}{4}, & e=\frac{J+|m|-1}{4}, & f=\frac{|m_2|-1}{2}. \end{array}$$

Let us mention an interesting fact that within the limit of a free motion in formula (32) for the interbasis coefficients $W_{Nlm}^{n_3}(\nu)$ instead of the hypergeometric function ${}_3F_2$ of the unit argument we have the function ${}_4F_3$; and instead of the Clebsch–Gordan coefficients of the SU(2) group, the Racah coefficients for one fourth values of the SU(1,1) group momentum. Analogous formulae arose in calculating the coefficients of transition between different hyperspherical systems of coordinates (in the formalism of "trees") and have been analysed in [32]. In our case, this fact allows an alternative calculation of the integral $A_{N|m|n_3}^{l}(\nu)$ in formula (28) at $\nu=0$.

5 Elliptic bases

The oblate elliptic system of coordinates (known as the elliptic–cylindrical I) has the form

$$\begin{array}{ll} q_1 = R\mathrm{sn}(\mu,k)\mathrm{dn}(\nu,k')\cos\phi, & q_2 = R\mathrm{sn}(\mu,k)\mathrm{dn}(\nu,k')\sin\phi, \\ q_3 = R\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k'), & q_0 = R\mathrm{dn}(\mu,k)\mathrm{sn}(\nu,k'). \end{array}$$

$$-K \le \mu \le K, -2K' \le \nu \le 2K', 0 \le \phi < 2\pi,$$

where the elliptic Jacobi functions of the variables α and β have the moduli k and k', respectively, $k^2+k'^2=1$, and K and K' are the complete elliptic integrals.

For the potential V in the elliptic system of coordinates we have

$$V(\mu,\nu) = \frac{1}{2}M\omega^2R^2\left[\frac{1}{\mathrm{dn}^2\mu\mathrm{sn}^2\nu}-1\right]. \tag{33}$$

Choosing the wave function Ψ in the form

$$\Psi(\mu,\nu,\varphi) = \psi_1(\mu)\psi_2(\nu)\frac{e^{im\varphi}}{\sqrt{2\pi}}, \qquad m\in\mathbf{Z}, \tag{34}$$

after the separation of variables in the Schrödinger equation (33) we arrive at two ordinary differential equations

$$\begin{aligned} \frac{d^2\psi_1}{d\mu^2} + \frac{\mathrm{cn}\mu\mathrm{dn}\mu}{\mathrm{sn}\mu}\frac{d\psi_1}{d\mu} - \left[\left(\frac{2MER^2}{\hbar^2}+\frac{M^2\omega^2R^4}{\hbar^2}\right)k^2\mathrm{sn}^2\mu \right. &+ \left.\frac{m^2}{\mathrm{sn}^2\mu} - \frac{M^2\omega^2R^4}{\hbar^2}\frac{k'^2}{\mathrm{dn}^2\mu}\right]\psi_1 \\ &= -\lambda_q(k;R)\psi_1, \qquad (35) \\ \frac{d^2\psi_2}{d\nu^2} - k'^2\frac{\mathrm{sn}\nu\mathrm{cn}\nu}{\mathrm{dn}\nu}\frac{d\psi_2}{d\nu} + \left[\left(\frac{2MER^2}{\hbar^2}+\frac{M^2\omega^2R^4}{\hbar^2}\right)\mathrm{dn}^2\nu \right. &+ \left.\frac{k^2m^2}{\mathrm{dn}^2\nu} - \frac{M^2\omega^2R^4}{\hbar^2}\frac{1}{\mathrm{sn}^2\nu}\right]\psi_2 \\ &= +\lambda_q(k;R)\psi_2, \qquad (36) \end{aligned}$$

where the quantum number q enumerates the elliptic separation constant $\lambda_q(k;R)$. Excluding from the equations (35) and (36) energy E, we come to the following operator

$$\begin{aligned}\Lambda &= \frac{1}{k^2\mathrm{sn}^2\mu - \mathrm{dn}^2\nu}\left[\mathrm{dn}^2\nu\frac{\partial^2}{\partial\mu^2} + k^2\mathrm{sn}^2\mu\frac{\partial^2}{\partial\nu^2} + \frac{\mathrm{cn}\mu\mathrm{dn}\mu}{\mathrm{sn}\mu}\mathrm{dn}^2\nu\frac{\partial}{\partial\mu} - k'^2k^2\frac{\mathrm{sn}\nu\mathrm{cn}\nu}{\mathrm{dn}\nu}\frac{\partial}{\partial\nu}\right] \\ &+ \frac{M^2\omega^2R^4}{\hbar^2}\frac{\mathrm{dn}^2\nu + k^2\mathrm{sn}^2\mu - 1}{\mathrm{dn}^2\mu\mathrm{sn}^2\nu} - \frac{\mathrm{dn}^2\nu + k^2\mathrm{sn}^2\mu}{\mathrm{dn}^2\nu\mathrm{sn}^2\mu}\frac{\partial^2}{\partial\varphi^2} \\ &= (1-k^2)L^2 - k^2R^2D_{33} + k^2L_3^2 + (k'^2)\frac{M^2\omega^2R^4}{\hbar^2} + k^2\frac{2MR^2}{\hbar^2}H, \end{aligned} \tag{37}$$

whose eigenvalues are $\lambda_q(k;R)$, and eigenfunctions are given by expression (34). Let us introduce a new operator according to

$$\begin{aligned}\Im^{\mathrm{obl.}} &= \frac{1}{1-k^2}\left\{\Lambda + (1-k^2)\frac{M^2\omega^2R^4}{\hbar^2} - k^2\frac{2MR^2}{\hbar^2}H + k^2L_3^2\right\} \\ &= L^2 - aR^2D_{33}, \end{aligned} \tag{38}$$

where $a = k^2/(1-k^2) \in [0,\infty)$. As is known [27], a transition from the oblate to the prolate elliptic system of coordinates on the sphere can be obtained under the transformation $k \to ik/k'$, $k' \to 1/k'$. In this case, the operator (38) for the prolate system of coordinates has the form

$$\Im^{\mathrm{prl.}} = L^2 + k^2R^2D_{33}. \tag{39}$$

The latter formula allows one to describe both the elliptic systems of coordinates uniquely with the use of the operator

$$\Im = L^2 - aR^2D_{33}, \tag{40}$$

where $a \in [-1,\infty)$. For positive a we have the oblate system; and for $a \in [-1,0]$, the prolate elliptic system of coordinates.

5.1 Expansion of elliptic bases over the hyperspherical and cylindrical ones

Thus, we have seen in the previous sections that all three oscillator bases: hyperspherical, cylindrical and both the elliptic ones, are eigenfunctions of three complete sets of operators $\{H, L^2, L_z^2\}$, $\{H, D_{33}, L_z^2\}$ and $\{H, \Im, L_z^2\}$ so that

$$L^2\Psi_{Nlm}(\chi,\vartheta,\varphi) = l(l+1)\Psi_{Nlm}(\chi,\vartheta,\varphi), \tag{41}$$

$$D_{33}\Psi_{Nn_3m}(\varphi_1,\alpha,\varphi_2) = (n_3+\nu+1)^2\Psi_{Nn_3m}(\varphi_1,\alpha,\varphi_2), \tag{42}$$

$$\Im\Psi_{nqm}(\mu,\nu,\varphi) = \lambda_q(a;R)\Psi_{nqm}(\mu,\nu,\varphi). \tag{43}$$

The operator equations (41)-(43) allow us to construct elliptic bases of the isotropic oscillator on the sphere as a superposition over the hyperspherical and cylindrical bases.

Now, let us write the sought expansions:

$$\Psi_{Nqm}(\mu,\nu,\varphi) = \sum_{l=|m|,|m|+1}^{N} T_{Nqm}^{l}(a;R)\Psi_{Nlm}(\chi,\vartheta,\varphi), \tag{44}$$

$$\Psi_{Nqm}(\mu,\nu,\varphi) = \sum_{n_3=0,1}^{N-|m|} U_{Nqm}^{n_3}(a;R)\Psi_{Nn_3m}(\varphi_1,\alpha,\varphi_2). \tag{45}$$

Consider the expansion (44). Substituting (44) into the operator equation (42), we find

$$\frac{1}{aR^2}\left\{l(l+1)-\lambda_q(a;R)\right\}T_{Nqm}^{l} = \sum_{l'=0}^{N} T_{Nqm}^{l'}(D_{33})_{ll'}, \tag{46}$$

where

$$(D_{33})_{ll'} = \int \Psi_{Nlm}^{*}(D_{33})\Psi_{Nl'm}d\Omega. \tag{47}$$

To calculate the integral (47) we use the expansion of the spherical basis over the cylindrical one and equation (42) for the eigenfunctions of the operator D_{33}. As a result, we come to the following expression for $(D_{33})_{ll'}$:

$$(D_{33})_{ll'} = \sum_{n_3}^{N-|m|} W_{Nlm}^{n_3}W_{Nl'm}^{n_3}(n_3+\nu+1)^2, \tag{48}$$

Then, using the three-term recurrence relations for the Racah coefficients [31]

$$cB_c\left\{\begin{matrix} a,b,c+1 \\ d,l,f \end{matrix}\right\} + (c+1)B_{c-1}\left\{\begin{matrix} a,b,c-1 \\ d,l,f \end{matrix}\right\} + (2c+1)A_c\left\{\begin{matrix} a,b,c \\ d,l,f \end{matrix}\right\} = 0, \tag{49}$$

where

$$\begin{aligned} B_c &= \sqrt{(a+b+c+2)(-a+b+c+1)(a-b+c+1)(a+b-c)} \\ &\times \sqrt{(d-l+c+1)(d+l-c)(d+l+c+2)(-d+l+c+1)}, \end{aligned} \tag{50}$$

$$\begin{aligned} A_c &= [a(a+1)-b(b+1)][d(d+1)-l(l+1)] + c(c+1)[a(a+1) \\ &+ b(b+1)+d(d+1)+l(l+1)-c(c+1)] - 2c(c+1)f(f+1) \end{aligned} \tag{51}$$

and the orthogonality property

$$\sum_{n_3=0,1}^{N-|m|} W_{Nlm}^{n_3}W_{Nl'm}^{n_3} = \delta_{ll'}, \tag{52}$$

we have

$$(D_{33})_{ll'} = -\frac{16B_{l-2}}{(2l-1)(2l+1)}\delta_{l-2,l'} + C_l\delta_{ll'} - \frac{16B_l}{(2l+1)(2l+3)}\delta_{l+2,l'}$$

where

$$
\begin{aligned}
B_l &= \frac{1}{16}\sqrt{(l-|m|+1)(l-|m|+2)(l+|m|+1)(l+|m|+2)(N+l+3)(N-l)} \\
&\times \sqrt{(N+l+2\nu+4)(N-l+2\nu+1)}, \qquad (53) \\
C_l &= \frac{1}{8}\Big\{4(N+1)(N+3)+2(2|m|^2-1)+4\nu(2N+2\nu+5)-(2l-1)(2l+3) \\
&- \frac{(4|m|^2-1)(2N+3)(2N+5+\nu)}{(2l-1)(2l+3)}\Big\} \qquad (54)
\end{aligned}
$$

Substituting the matrix element (53) into (46), we finally arrive at the three-term recurrence relation

$$
\begin{aligned}
&\frac{16}{(2l-1)(2l+1)} B_{l-2} T^{l-2}_{Nqm}(a;R) + \left\{\frac{1}{aR^2}\left[l(l+1)-\lambda_q(a;R)\right] - C_l\right\} T^l_{Nqm}(a;R) \\
&\qquad + \frac{16}{(2l+1)(2l+3)} B_l T^{l+2}_{Nqm}(a;R) = 0, \qquad (55) \\
&\qquad T^{-1}_{Nqm}(a;R) = T^{-2}_{Nqm}(a;R) = 0.
\end{aligned}
$$

for the expansion coefficients $T^l_{Nqm}(a;R)$. The recurrence relation (55) is a system of homogeneous equations which has to be solved with the normalization condition

$$
\sum_{l=|m|,|m|+1}^{N} |T^l_{Nqm}(a;R)|^2 = 1.
$$

Eigenvalues of the elliptic separation constant $\lambda_q(a;R)$ are calculated from the condition for the determinant of the system of homogeneous equations to be equal to zero (55).

Consider now the expansion (45) of the elliptic basis over the cylindrical one. In a similar way, like in the calculation of the expansion coefficients (44), we get

$$
\left\{\lambda_q(k;R) + aR^2(n_3+\nu+1)^2\right\} U^{n_3}_{Nqm}(k;R) = \sum_{n_3'=0,1}^{N-|m|} U^{n_3'}_{Nqm}(k;R)(L^2)_{n_3 n_3'}, \qquad (56)
$$

where

$$
(L^2)_{n_3 n_3'} = \int \Psi^*_{Nn_3m} L^2 \Psi_{Nn_3'm} d\Omega. \qquad (57)
$$

The integral in (57) can be calculated if one uses the expansion of the cylindrical basis over the spherical one and then the symmetry property of the Racah coefficients

$$
\left\{\begin{matrix} a, b, c \\ d, l, f \end{matrix}\right\} = \left\{\begin{matrix} a, l, f \\ d, b, c \end{matrix}\right\} \qquad (58)
$$

and the three-term recurrence relation (49). As a result of simple calculations we obtain the expression ($U^{n_3}_{Nqm} \equiv U_{n_3}$)

$$
\tilde{B}_{n_3} U_{n_3+2} + \left\{\tilde{C}_{n_3} - \lambda_q(k;R) - aR^2(n_3+\nu+1)^2\right\} U_{n_3} + \tilde{B}_{n_3-2} U_{n_3-2} = 0, \qquad (59)
$$

where

$$\tilde{B}_{n_3} = \frac{1}{4}\sqrt{\frac{(n_3+2\nu+2)(n_3+2)(n_3+1)(n_3+2\nu+3)(N+|m|+n_3+2\nu+4)}{(n_3+\nu+1)(n_3+\nu+2)^2(n_3+\nu+3)}}$$
$$\times \sqrt{(N+|m|-n_3)(N-|m|-n_3)(N-|m|+n_3+2\nu+4)},$$
$$\tilde{C}_{n_3} = \frac{1}{2}\Big\{(N+2)^2+\nu(2N+2\nu+5)+(|m|^2-2)-(n_3+\nu)(n_3+\nu+2)$$
$$- \frac{\nu(\nu+1)(N+|m|+\nu+2)(N-|m|+\nu)}{(n_3+\nu)(n_3+\nu+2)}\Big\}.$$

As in the previous case, a homogeneous system of equations should be solved together with the normalization condition

$$\sum_{n_3=0,1}^{N-|m|} |U_{Nqm}^{n_3}(a;R)|^2 = 1,$$

and the separation constant can again be determined from the condition of equality to zero of the corresponding determinant of the homogeneous system of equations (59).

In conclusion, we would like to mention that in the limit of the free motion $\nu \to 0$, the three-term recurrence relations (55) and (59) transform into those for the expansion coefficients of the elliptic basis over cylindrical and spherical ones, which have been obtained in the work [27].

6 Conclusion

In the present paper, a first step has been made to completely study the Schrödinger equation and to calculate the coefficients of various interbasis expansions for the potential of the isotropic oscillator on the three-dimensional sphere in different orthogonal systems of coordinates. We have calculated interbasis expansions for the "sphere–cylinder" transition and also constructed solutions of the Schrödinger equation in both the elliptic systems of coordinates as expansion over the spherical and cylindrical bases. In contrast with the "sphere–cylinder" transition, in which the transformation coefficients are expressed through the generalised hypergeometric functions ${}_4F_3$ of the unit argument or through the Racah coefficients, extended over their indices into the region of arbitrary real values, the transition coefficients for the elliptic bases are defined by the three-term recurrence relations and cannot be written down explicitly.

In this paper we have considered the sphero-conic and ellipsoidal bases of the isotropic oscillator. In separating variables in the Schrödinger equation for the sphero-conic system of coordinates we arrive at the quasiradial equation (13) considered in Sect. 3 and two standard Lame equations that are derived in separating variables for the Helmholtz equation on the two-dimensional sphere. As a result, the solution of the Schrödinger equation is

$$\Psi(\chi,\alpha,\beta;R) = Z_{Nl}^{\nu}(\chi)\,\Lambda_{l\lambda}(\alpha)\Lambda_{l\lambda}(\beta),$$

where the function $Z^{\nu}_{Nl}(\chi)$ is determined by expression (16), and the explicit form of the Lame polynomials $\Lambda_{l\lambda}(\alpha)$ can be found in [28, 36].

For the ellipsoidal basis of the isotropic oscillator, after separation of variables in the Schrödinger equation we obtain three identical equations containing two ellipsoidal separation constants λ_1, λ_2. This means that in contrast with the cases considered we deal with the two-parametric spectral problem. Application of the method of constructing solutions of the Schrödinger equation as expansion over simpler bases leads to two many-termed recurrence relations determined by the cubic matrix. Obviously, the simplest way of constructing an ellipsoidal basis consists in applying the Niven method [37] allowing one to write down a solution to the Schrödinger equation in terms of zeroes of the wave function and reduce the problem to a system of relevant nonlinear equations. This kind of investigation is beyond the scope of the present paper and will be carried out elsewhere.

In conclusion, we would like to thank L.G. Mardoyan and V.M. Ter-Antonyan for useful interesting discussions.

References

[1] Schrödinger E., *Proc. of the Royal Irish.Acad.* 1940, **A46**, p. 9; p. 183; 1941, **A47**, p. 53.

[2] Infeld L. and Schild A., *Phys.Rev.* 1945, **67**, p. 121.

[3] Stevenson A.F., *Phys.Rev.* 1941, **59**, p. 842.

[4] Bessis N. and Bessis G., *J.Phys.* 1979, **A12**, p. 1991.

[5] Izmest'ev A.A., *Yad. Fiz.* 1991, **52**, p. 1697; 1991, **53**, p. 1402 (in Russian).

[6] Bogush A.A., Otchik V.S., *J.Phys.* 1997, **A30**, p. 559.

[7] Nishino Y., *Math. Japon.* 1972, **17**, p. 59.

[8] Higgs P.W., *J.Phys.* 1979, **A12**, p. 309.

[9] Leemon H.I., *J.Phys.* 1979, **A12**, p. 489.

[10] Kurochkin Y.A., Otchik V.S., *DAN BSSR* 1979, **XXIII**, p. 987 (in Russian).

[11] Bogush A.A., Kurochkin Y.A., Otchik V.S., *DAN BSSR* 1980, **XXIV**, p. 19 (in Russian).

[12] Bogush A.A., Otchik V.S., Red'kov V.M., *Vestnik AN BSSR* 1983, **3**, p. 56 (in Russian).

[13] Bogush A.A., Red'kov V.M., *Quantum – Mechanical Kepler problem in constant curvature spaces*, Preprint 298, *IN AN BSSR* 1983 (in Russian).

[14] Granovsky Ya.A., Zedanov A.S., Luzenko I.M., *Teor.Mat.Fiz.* 1992, **91**, p. 207 (in Russian).

[15] Granovsky Ya.A., Zedanov A.S., Luzenko I.M., *Teor.Mat.Fiz.* 1992, **91**, p. 396 (in Russian).

[16] Barut A.O., Inomata A. and Junker G., *J.Phys.* 1987, **A20**, p. 6271.

[17] Barut A.O., Inomata A. and Junker G., *J.Phys.* 1990, **A23**, p. 1179.

[18] Grosche C., *Ann. Phys. (N.Y.)* 1990, **204**, p. 208.

[19] Grosche C., Pogosyan G.S. and Sissakian A.N., *Fortschritte der Physik*, 1995, **43**, p. 523.

[20] Grosche C., Pogosyan G.S. and Sissakian A.N., *Phys.Part.Nucl.* 1996, **27**(3), p. 244.

[21] Grosche C., Pogosyan G.S. and Sissakian A.N., *Phys.Part.Nucl.* 1997, **28**(5).

[22] Olevskii, *Mat. Sb.* 1950, **27**, p. 379 (in Russian).

[23] Demkov Yu.N., *JETP* 1965, **36**, p. 88 (in Russian).

[24] Kadyshevsky V.G., *Dok.Akad.Nauk. SSSR* 1962, **147**, p. 588 (in Russian).

[25] Pogosyan G.S. and Sissakian A.N., *Turkish J.Phys.* 1997, **21**, p. 515.

[26] Izmest'ev A.A., Pogosyan G.S., Sissakian A.N. and Winternitz P., *J.Phys.* 1996, **A29**, p. 5949.

[27] Grosche C., Karayan Kh.H., Pogosyan G.S. and Sissakian A.N., *J.Phys.* 1997, **A30**, p. 1629.

[28] Bateman H., Erdélyi A., *Higher transcendental functions*, New York–Toronto–London, MC Graw–Hill Book Company, INC, 1953.

[29] Bailey W.N., *Generalized hypergeometric series*, Cambridge Tracts, No 32, Cambridge, 1935.

[30] Prudnikov A.P., Brichkov Yu.A., Marichev O.I., *Integrals and series. Special functions.*, Nauka, Moskva, 1986 (in Russian).

[31] Varshalovich D.A., Moskalev A.N. and Khersonskii V.K., *Quantum theory of angular momentum*, World Scientific, Singapore, 1988.

[32] Kuznetsov G.I., Smorodinsky Ya.A., *Yad. Fiz.* 1977, **25**, p. 447. (in Russian).

[33] Pogosyan G.S., Ter–Antonyan V.M., *Conversion Factors for Cartesian, Polar and Cylindrical wave functions of Isotropic Oscillator.* Communication JINR 1978, P2-11962, Dubna (in Russian).

[34] Szego G., *Orthogonal Polynomials*, Amer.Math.Society, New York, 1959.

[35] Flügge S., *Practical Quantum Mechanics*, Springer–Verlag, Berlin–Heidelberg–New York, 1971.

[36] Patera J. and Winternitz P., *J.Math.Phys.* 1973, **14**, p. 1130.

[37] Whittaker E.T., Watson G.N., *A course of modern analysis*, Cambridge University Press 1927, **2**.